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Solution of a System of Integro-Differential Equations by Homotopy Perturbation Method

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Abstract

This paper is an attempt to use the homotopy perturbation method (HPM) to solve equations containing integro-differential systems. To show the ability of the method an example is also developed. The results have revealed high ability of the present method as compared to Adomian's decomposition method regarding the corresponding errors.

1. Introduction

Most integro-differential equations such as engineering problems of damper and spring are nonlinear and difficult to solve. There are some nonlinear methods for solving the problems like homotopy analysis method (HAM) [1-3] and Adomian's decomposition method (ADM) [4]. However, these methods fail to develop a simple way to control the convergence region and the rate of given approximate series. Among the developed methods homotopy perturbation method (HPM) is rapid convergent method [5-8] which can present the closest approximation to exact solution. Additionally, the mentioned method has a high ability in solving integro-differential equations [9]. A system of integro-differential Equations can be considered in general form as [10]

$$df_1(t)/dt = H_1\left(t, f_1(t), \dots, f_n(t), \int_0^t K_1\big(t, s, f_1(s), \dots, f_n(s)\big)ds\right),$$

$$df_2(t)/dt = H_2\left(t, f_1(t), \dots, f_n(t), \int_0^t K_2\big(t, s, f_1(s), \dots, f_n(s)\big)ds\right),$$

$$\vdots$$

$$df_n(t)/dt = H_n\left(t, f_1(t), \dots, f_n(t), \int_0^t K_n\big(t, s, f_1(s), \dots, f_n(s)\big)ds\right)$$

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where each equation represents the first derivative of one of unknown functions as a mapping involving the independent variable t and n unknown functions $f_1, f_2, \ldots, f_n$ which have appeared partly under the integral. For economy of writing, a system of integro-differential equation is in the form

$$df_i(t)/dt = H_i\left(t, f_1(t), \ldots, f_n(t), \int_0^t K_i\big(t, s, f_1(s), \ldots, f_n(s)\big)ds\right) \tag{1}$$

2. The basic concepts of homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation [2, 3, 5, 11, 12]

$$A(u) - f(r) = 0, r \in \Omega \tag{2}$$

considering the boundary conditions of

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma \tag{3}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

The operator A can be divided into two parts of L and N, where L is the linear part, while N is a nonlinear one. Eq. (2) can, therefore, be rewritten as

$$L(u) + N(u) - f(r) = 0 \tag{4}$$

Using the homotopy technique, we construct a homotopy as $v(r,p)$: $\Omega \times [0,1] \rightarrow \Re$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad P \in [0,1], \quad r \in \Omega \tag{5}$$

where $P \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (5) which satisfies the boundary conditions. Obviously, considering Eq. (5) we will have

$$H(v,p) = L(v) - L(u_0) = 0 \tag{6}$$

$$H(v,0) = A(v) - f(r) = 0 \tag{7}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to HPM, we can first use the embedding parameter p as a “small parameter”, and assume that the solution of Eq. (5) can be written as a power series in p

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{8}$$

Setting $p = 1$, results in the approximate solution of Eq. (7) as

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{9}$$

HPM overcomes the limitations of the traditional perturbation methods. However, this technique can have full advantages of the traditional perturbation techniques [13].

The series (9) is convergent for most cases. However, the convergence rate depends on the nonlinear operator $A(v)$. The following opinions are suggested by He [7, 8] as

- The second derivative of $N(v)$ with respect to $\boldsymbol{v}$ must be small because the parameter p may be relatively large, i.e. $p \to 1$.
- The norm of $L^{-1}\,\partial N/\partial v$ must be smaller than one so that the series converges.

3. Application

In this part, we present an example in the form

$$\begin{aligned} u'(x) &= 1 + x + x^2 - v(x) - \int_0^x \big(u(t) + v(t)\big) d(t) \\ v'(x) &= -1 - x + u(x) - \int_0^x \big(u(t) - v(t)\big) d(t) \end{aligned} \tag{10}$$

with the following exact solutions

$$u(x) = x + e^x, \quad v(x) = x - e^x \tag{11}$$

In first step, we calculated the derivatives of the system and for integral part we are using the Leibniz law. Based on the procedure, the system will be

$$\begin{aligned} & u'' + u + v' + v - 1 - 2x = 0 \\ & v'' - v - u' + u + 1 = 0 \end{aligned} \tag{12}$$

Applying HPM to the system will result in

$$\begin{aligned} H(u,p) &= (1-p)[u'' + u] + p[u'' + u + v' + v - 1 - 2x] \\ H(v,p) &= (1-p)[v'' - v] + p[v'' - v - u' + u + 1] \end{aligned} \tag{13}$$

where u and v are

$$u = u_0 + pu_1 + p^2u_2$$
$$v = v_0 + pv_1 + p^2v_2 \tag{14}$$

Applying the initial conditions and replacing them into general equation different terms of *u* and *v* will be obtained:

$$u(x) = -\frac{5}{2}sin(x) + \frac{3}{2}e^x + 2x + \frac{1}{2}(-1 + 2x)cos(x) - \frac{1}{2}\sin(x)$$
$$v(x) = -\frac{5}{4}e^x + \frac{1}{4}e^{-x} - \frac{1}{2}sin(x) + 2x \tag{15}$$

The same procedure is applicable for more terms of *u(x)* and *v(x)* in order to get more exact approximations of variables.

4. Results and Discussions

The results of HPM contained three terms approximations to the solutions. These results have been summarized in Table 1. To more consciously evaluate the results, we also considered approximations to the solutions obtained from Adomian's decomposition method [14]. The results of this method are also summarized in Table 1. It is important to point out that Adomian's decomposition method has used four iterations, while we applied three iterations. Despite this fact, the results showed that there are no considerable differences between the two methods in terms of *v*; however HPM results are more reliable. In the case of *u*, the difference is not considerable compared to that of *v*.

Table 1. The comparison between ADM and HPM with the related errors

x	$e(u(x))[Adomian]$	$e(u(x))[Homotopy]$	$e(v(x))[Adomian]$	$e(v(x))[Homotopy]$
0	0	0	0	0
0.2	0.141146E-3	0.2667E-5	0.71716E-4	0.6602E-4
0.4	0.2376726E-2	0.85333E-4	0.1218010E-2	0.1049939E-2
0.6	0.12594035E-1	0.648028E-3	0.64663277E-2	0.5211489E-2
0.8	0.41449813E-1	0.2731036E-2	0.21150504E-1	0.1619093E-1
1	0.104879113	0.8336089E-2	0.52816659E-1	0.3771549E-1

In order to have a simple comparison to the solutions obtained by HPM, the solutions are shown graphically in Figures 1 and 2.

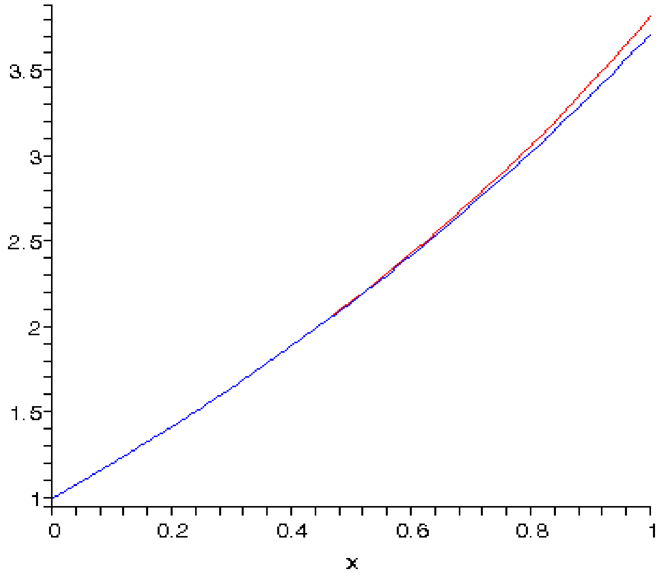

Figure 1. The obtained solution by HPM and ADM for *u(x)*

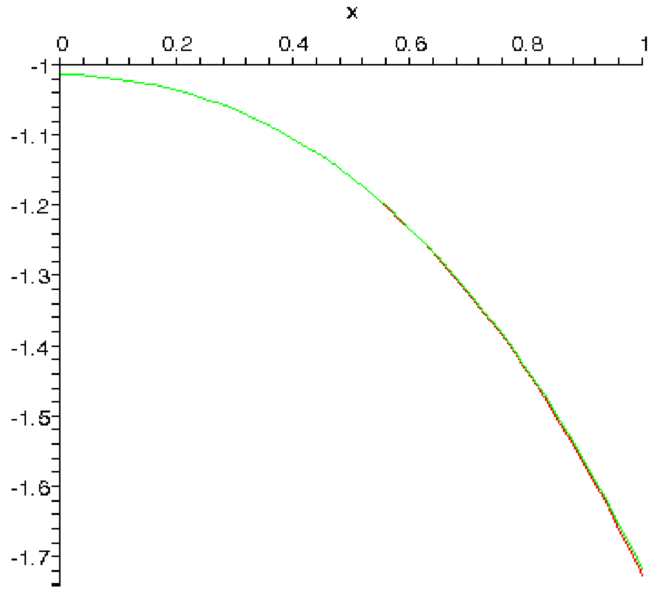

Figure 2. The obtained solution by HPM and ADM for *v(x)*

5. Conclusions

Homotopy perturbation method (HPM) has been known as a powerful device for solving many functional equations as algebraic equations, ordinary and partial differential equations, integral equations, etc. Here, we used this method for solving systems of integro-differential equations. As it was shown, this method has the ability of solving systems of linear integral- differential equations and can be found widely applicable in engineering and physics problems.

References

[1] Liao S. J.: Beyond Perturbation: Introduction to the Homotopy Analysis Method. Chapman & Hall/CRC Press, Boca Raton (2003).

[2] He J. H.: Comparison of homotopy perturbation and homotopy analysis method. Applied Mathematics and Computation, **156**, 527–539 (2004).

[3] He J. H.: The homotopy perturbation method for nonlinear oscillators with discontinuities. Applied Mathematics and Computation, **151**, 287–292 (2004).

[4] Adomian G.: Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic, Dordrecht (1994).

[5] Cveticanin L.: Homotopy perturbation method for pure nonlinear differential equation. Chaos, Solitons & Fractals, **30**, 1221–1230 (2005).

[6] He J. H.: Homotopy perturbation method: a new nonlinear analytic technique. Applied Mathematics and Computation, **135**, 73–79 (2003).

[7] He J. H.: Application of homotopy perturbation method to nonlinear wave equations. Chaos, Solitons & Fractals, **26**, 695–700 (2005).

[8] He J. H.: Homotopy perturbation method for bifurcation of nonlinear problems. International Journal of Nonlinear Sciences and Numerical Simulation, **6**, 207–208 (2005).

[9] El-Shahed M.: Application of He's homotopy perturbation method to Volterra's integro-differential equation. International Journal of Nonlinear Sciences and Numerical Simulation, **6**, 163–168 (2005).

[10] Ganji D. D.: Application of homotopy perturbation method to the second kind of nonlinear integral equations. Physics Letters A, **371**, 20–25 (2007).

[11] He J. H.: Homotopy perturbation technique. Computer Methods in Applied Mechanics and Engineering, **178**, 257–262 (1999).

[12] He J. H.: Homotopy perturbation method for solving boundary value problems. Physics Letters A, **350**, 87–88 (2006).

[13] He J. H.: A coupling method of homotopy and perturbation technique for nonlinear problems. International Journal of Nonlinear Mechanics, **35**, 527–539 (2000).

[14] Baizar J.: Solution of System of Integro-Differential Equations by Adomian's decomposition method. Applied Mathematics and Computation, **168**, 1232–1238 (2003).